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A NON-LINEAR VERSION OF THE AMIR–LINDENSTRAUSS METHOD

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ABSTRACT
A non-linear version of the Amir–Lindenstrauss method of projections for weakly compactly generated Banach spaces is proved, that implies immediately the Benyamini–Rudin–Wage result on continuous images of Eberlein compact spaces.

A Banach space is called *weakly compactly generated* (W.C.G.) if it is generated by a weakly compact subset. The theorem of Amir–Lindenstrauss [1], fundamental for the structure of such Banach spaces, states that for every W.C.G. Banach space X there is a set Γ and a bounded one-to-one linear operator $T: X \to c_0(\Gamma)$; from this theorem, it follows that every *Eberlein compact space* (i.e., every weakly compact subset of some Banach space) is homeomorphic to a weakly compact subset of $c_0(\Gamma)$. A problem in the theory of Eberlein compact sets, not resolved either by the original Amir–Lindenstrauss result, or by a characterization of Eberlein compact sets given by Rosenthal in [6] (based on the Amir–Lindenstrauss theorem), was whether the continuous image of an Eberlein compact space is an Eberlein compact space. The original proof of this result, by Y. Benyamini, M. E. Rudin and M. Wage [2], is rather involved (a simpler proof was found subsequently by E. Michael and M. E. Rudin [5]).

The main result of this paper, given in Theorem 2.9, is to prove a non-linear version of the Amir–Lindenstrauss method. It is interesting that Dowker's theorem on the possibility of interpolating a continuous function between a smaller upper-semicontinuous function and a larger lower-semicontinuous function on a normal countably paracompact space plays a crucial role in the proof of this result.

The usefulness of our result is seen in the fact that it implies immediately the Benyamini–Rudin–Wage theorem (Corollary 2.11).

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§1. We will make use of the following facts.

1.1. PROPOSITION ([1], lemma 1). *Let* $\|\cdot\|$, $|||\cdot|||$ *be two norms on* $\mathbf{R}^n$. *Then there exist* $z_1, z_2, \cdots, z_n \in \mathbf{R}^n$ *such that for every* $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n$ *we have*

$$\left\| \sum_{i=1}^n \lambda_i \frac{z_i}{\|z_i\|} \right\| \geqq \frac{1}{n} |\lambda|, \qquad \textit{and}$$

$$\left|\left|\left| \sum_{i=1}^n \lambda_i \frac{z_i}{|||z_i|||} \right|\right|\right| \geqq \frac{1}{n} |\lambda|,$$

where $|\lambda| = (\lambda_1^2 + \lambda_1^2 + \cdots + \lambda_1^2)^{1/2}$.

1.2. PROPOSITION. *Let B be a norm bounded subset of* $c_0(\Gamma)$. *Then the weak topology of B concides with the topology of pointwise convergence.*

1.3. THEOREM (Grothendieck). *Let* Ω *be a compact space and B a bounded subset of* $C(\Omega)$. *Then B is weakly compact if and only if B is compact for the topology of pointwise convergence.*

(A proof of this theorem is given in [3] (p. 156).)

§2. The main result is the (non-linear) Theorem 2.9. Most of the lemmas used to prove it are refinements of the corresponding lemmas in the Amir–Lindenstrauss paper [1]; consequently, we try to avoid repetition, in the proofs below, concentrating only on the new aspects of the proof.

2.1. LEMMA. *Let X be a linear space with two norms* $\|\cdot\|$, $|||\cdot|||$ *such that* $\|x\| \geqq |||x|||$ *for every* $x \in X$. *Also, let Y be a normed space and* $R: X \to Y$ *be a* $|||\cdot|||$*-bounded linear operator. Let also* $\varepsilon > 0$, *n a natural number, B a finite dimensional subspace of X, and* $f_1, \cdots, f_m \in (X, \|\cdot\|)^*$ *of* $\|\cdot\|$ *norm* 1. *Then there is a separable subspace C of X that contains B such that if Z is a subspace of X with* $B \subset Z$ *and* $\dim(Z/B) = n$ *then there is a linear operator* $T_Z : Z \to C$ *so that*

(i) $\|T\| \leqq 1 + \varepsilon$, $|||T||| \leqq 1 + \varepsilon$,

(ii) $T(b) = b$ *for* $b \in B$,

(iii) $|f_k(z) - f_k(T(z))| \leqq \varepsilon \|z\|$ *for* $k = 1, 2, \cdots, m$, *and*

(iv) $\|RT(z)\| \leqq \varepsilon + \|R(z)\|$ *for* $z \in Z$ *with* $|||z||| \leqq 1$.

PROOF. We set $\beta = \max\{|||b|||,\ b \in B,\ \|b\| = 1\}$. Let P be a $\|\cdot\|$-bounded projection of X into B. Then we have

$$|||P||| \leqq \beta \|P\|.$$

Let $M > \max\{14n^2\|P\| \cdot 1/\varepsilon, (2\|R\| + 4\sqrt{n}\|R\| + 1) \cdot 1/\varepsilon\}$. We choose $b_1, \cdots,$

$b_p \in B$ such that for every $b \in B$ with $\|b\| \leqq M$ there is $1 \leqq h \leqq p$ such that $|||b - b_h||| < 1/M$. We set

$$\Sigma = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in R^n : |\lambda| = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2} \leqq n(1 + \beta \|P\|)\},$$

and choose $\lambda^1, \cdots, \lambda^q \in \Sigma$ such that for every $\lambda \in \Sigma$ there is $1 \leqq j \leqq q$ such that $|\lambda - \lambda^j| < 1/M$.

On $(X \backslash \{0\})^n$ we consider the following $3pq + (m+4)n$ real functions:

$$\left\| b_h + \sum_{i=1}^n \lambda_i^j \frac{x_i}{\|x_i\|} \right\|, \quad \left|\left|\left| b_h + \sum_{i=1}^n \lambda_i^j \frac{x_i}{|||x_i|||} \right|\right|\right|, \quad f_k\left(\frac{x_i}{\|x_i\|}\right),$$

$$\|x_i\|, \quad |||x_i|||, \quad \log\|x_i\|, \quad \log|||x_i|||, \quad \left\| R\left(b_h + \sum_{i=1}^n \lambda_i^j \frac{x_i}{|||x_i|||}\right)\right\|,$$

$$(1 \leqq h \leqq p, 1 \leqq i \leqq n, 1 \leqq j \leqq q, 1 \leqq k \leqq m).$$

These functions define a function

$$\Phi : (X \backslash \{0\})^n \to \mathbf{R}^{3pq + (m+4)n}.$$

(We consider the supremum distance ρ on $\mathbf{R}^{3pq+(m+4)n}$.) The image of Φ is separable; hence there is a sequence $\Phi(x_1^1, \cdots, x_n^1)$, $\Phi(x_1^2, \cdots, x_n^2), \cdots$ such that for every $(z_1, \cdots, z_n) \in (X \backslash \{0\})^n$ there is t such that

$$\rho(\Phi(x_1^t, \cdots, x_n^t), \Phi(z_1, \cdots, z_n)) < 1/M.$$

Let C be the subspace of X generated by

$$B \cup \{x_i^t, i = 1, \cdots, n, t = 1, 2, \cdots\}.$$

Then C is separable.

Let Z be a subspace of X such that $B \subset Z$ and $\dim(Z/B) = n$. Then from Lemma 1.1 there are $z_1, \cdots, z_n \in (I - P)Z$ such that

$$\left\| \sum_{i=1}^n \lambda_i \frac{z_i}{\|z_i\|} \right\| \geqq \frac{1}{n}|\lambda|, \quad \left|\left|\left| \sum_{i=1}^n \lambda_i \frac{z_i}{|||z_i|||} \right|\right|\right| \geqq \frac{1}{n}|\lambda|$$

for every $\lambda = (\lambda_1, \cdots, \lambda_n)$.

We choose $x_1, \cdots, x_n \in C$ such that

$$\rho(\Phi(x_1, \cdots, x_n), \Phi(z_1, \cdots, z_n)) < 1/M$$

and define

$$T : Z \to C \quad \text{by} \quad T\left(b + \sum_{i=1}^n \lambda_i z_i\right) = b + \sum_{i=1}^n \lambda_i x_i.$$

The operator T satisfies conditions (i)–(iii) (cf. [1], lemma 2).

For (iv), let $z = b + \sum_{i=1}^{n} \lambda_i z_i / |||z_i|||$ with $|||z||| \leqq 1$. We have

$$\frac{1}{n}|\lambda| \leqq \left|\left|\left| \sum_{i=1}^{n} \lambda_i \frac{z_i}{|||z_i|||} \right|\right|\right|$$

$$\leqq |||I - P||| \cdot \left|\left|\left| b + \sum_{i=1}^{n} \lambda_i \frac{z_i}{|||z_i|||} \right|\right|\right| \leqq (1 + \beta) \cdot \|P\|)|||z|||,$$

hence $|\lambda| \leqq n(1 + \beta \|P\|)$, and consequently there is $1 \leqq j \leqq q$ with $|\lambda - \lambda^j| < 1/M$. Also $\|b\| \leqq |||b||| \leqq |||P||| \cdot |||z||| \leqq \beta \|P\| < M$, hence there is $1 \leqq h \leqq p$ such that $\|b - b_n\| < 1/M$.

Now, we have

$$\left\| R\left(b + \sum_{i=1}^{n} \lambda_i \frac{x_i}{\|z_i\|}\right)\right\| - \left\| R\left(b + \sum_{i=1}^{n} \lambda_i \frac{z_i}{\|z_i\|}\right)\right\| \leqq 2\|R\| \cdot |||b - b_h|||$$

$$+ \|R\| \cdot \left|\left|\left| \sum_{i=1}^{n} (\lambda_i - \lambda_i^j) \frac{x_i}{|||x_i|||} \right|\right|\right| + \left\| R\left(b_h + \sum_{i=1}^{n} \lambda_i^j \frac{x_i}{|||x_i|||}\right)\right\|$$

$$- \left\| R\left(b_h + \sum_{i=1}^{n} \lambda_i^j \frac{z_i}{\|z_i\|}\right)\right\| + \|R\| \left|\left|\left| \sum_{i=1}^{n} (\lambda_i - \lambda_i^j) \frac{z_i}{|||z_i|||} \right|\right|\right|$$

$$+ \|R\| \cdot \left|\left|\left| \sum_{i=1}^{n} \lambda_i \frac{x_i}{|||x_i|||} \left(\frac{|||x_i|||}{|||z_i|||} - 1 \right) \right|\right|\right|$$

$$\leqq (2\|R\| + \sqrt{n}\|R\| + 1 + \sqrt{n} \cdot \|R\| + 2\sqrt{n}\|R\|) \frac{1}{M} < \varepsilon.$$

Consequently, $\|RT(z)\| \leqq \varepsilon + \|R(z)\|$.

2.2. Lemma. *Let X be a linear space with two norms $\|\cdot\|$, $|||\cdot|||$, such that the $|||\cdot|||$-unit ball U is $\|\cdot\|$-weakly compact. Also, let Y be a normed space and $R : X \to Y^*$ a linear operator such that R is $|||\cdot|||$-bounded and $R\,|2U : 2U \to Y^*$ is $\|\cdot\|$-weakly-weakly* continuous (i.e., continuous with respect to the $\|\cdot\|$-weak topology that X defines on $2U$, and the weak* topology of Y^*). Also, let B be a finite dimensional subspace of X and $f_1, f_2, \cdots \in (X, \|\cdot\|)^*$. Then there is a linear operator $T : X \to X$ such that*

(i) *$T(X)$ is $\|\cdot\|$-separable,*

(ii) $\|T\| = |||T||| = 1$,

(iii) *$Tb = b$ for all $b \in B$,*

(iv) *$T^* f_k = f_k$ for $k = 1, 2, \cdots$, and*

(v) *$\|RT(x)\| \leqq \|R(x)\|$ for $x \in X$, with $|||x||| \leqq 1$.*

PROOF. We may assume that $\|x\| \leqq |||x|||$ for every $x \in X$ (since U is weakly compact and hence $\|\cdot\|$-bounded), and that $\|f_n\| = 1$ for $n = 1, 2, \cdots$.

We apply Lemma 2.1 for $\varepsilon = 1/n$ and $m = n$, and let C_n be the corresponding subspace of X.

Let C be the $\|\cdot\|$-closed subspace of X generated by $\bigcup_{n=1}^{\infty} C_n$. Then C is $\|\cdot\|$-separable.

For every subspace Z of X with $B \subset Z$ and $\dim(Z/B) = n$ we have a linear operator $T_Z : Z \to C$ such that

$$\|T_Z\| \leqq 1 + \frac{1}{n}, \qquad |||T_Z||| \leqq 1 + \frac{1}{n},$$

$$Tb = b \qquad \text{for } b \in B,$$

$$|f_k(z) - f_k(T_Z(z))| \leqq \frac{1}{n}\|z\| \qquad \text{for } z \in Z \text{ and } k = 1, 2, \cdots, n,$$

$$\|RT_Z(z)\| \leqq 1/n + \|R(z)\| \qquad \text{for } z \in Z, \quad |||z||| \leqq 1.$$

We extend T_Z to the whole space X by setting $T_Z(x) = 0$ for $x \in X \backslash Z$.

We consider the set $\mathscr{B}$ of all the finite dimensional subspaces of X and we note that $\mathscr{B}$ is directed under set inclusion. Then the family $\{T_Z \mid U : Z \in \mathscr{B}\}$ is a net in $(2U)^U$. From Tychonof's product theorem it follows that the space $(2U)^U$ with the product topology, i.e. with the topology induced by the $\|\cdot\|$-weak topology of $2U$, is compact. Consequently the net $(T_Z)_{Z \in \mathscr{B}}$ has a subnet, say $(T_\alpha)_{\alpha \in A}$, that converges pointwise to some $T \in (2U)^U$ in the $\|\cdot\|$-weak topology.

We extend T to the whole space X by the rule:

$$T(x) = \begin{cases} |||x|||\, T(x/|||x|||) & \text{if } x \in X, \quad x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is linear and satisfies the conditions (i)–(iv) (cf. [1], lemma 3). For (v), let $x \in U$ and n a natural number. Then we have

$$\|RT_Z(x)\| \leqq 1/n + \|R(x)\| \qquad \text{when } \dim(z/B) \geqq n.$$

Hence $\|RT_\alpha(x)\| \leqq 1/n + \|R(x)\|$ finally for all α, that is $|RT_\alpha(x)(y)| \leqq 1/n + \|R(x)\|$ for every $y \in Y$ with $\|y\| \leqq 1$.

We have $T(x) = \lim_{\alpha \in A} T_\alpha(x)$ (where the limit is taken with respect to the $\|\cdot\|$-weak topology) and $T_\alpha(x) \in 2U$. Consequently, by the $\|\cdot\|$-weak-weak* continuity of $R \mid 2U$, it follows that

$$|RT(x)(y)| = \lim_{\alpha \in A} |RT_\alpha(x)(y)| \leqq 1/n + \|R(x)\|.$$

Hence $\|RT(x)\| \leqq 1/n + \|R(x)\|$ for $n = 1, 2, \cdots$, i.e.,

$$\|RT(x)\| \leqq \|R(x)\|.$$

In the following, we consider X with the norm $\|\cdot\|$. If Z is a subspace of X we denote by $\chi(Z)$ the *density character of* Z (i.e. the least cardinality of a dense subset of Z) and if F is a subspace of X^* we denote by $\chi^*(F)$ the density character of F with respect to the weak* topology. Also, if ξ is an ordinal number we denote by $|\xi|$ the cardinality of ξ.

The next lemma is proved using Lemma 2.2, in a way similar to the proof of lemma 4 in [1]; the additional condition (v) is valid because of the weak-weak* continuity of $R \mid U$ and the weak*-lower-semicontinuity of the norm.

2.3. Lemma. *Let* X *and* $R : X \to Y^*$ *be as in Lemma* 2.2, m *an infinite cardinal,* Z *a subspace of* X *with* $\chi(Z)| \leqq m$ *and* F *a subspace of* X^* *with* $\chi^*(F)| \leqq m$. *Then there is a projection* $P : X \to X$ *such that*

(i) $\chi(P(X)) \leqq m$,

(ii) $\|P\| = |||P||| = 1$,

(iii) $P(z) = z$ *for* $z \in Z$,

(iv) $P^*f = f$ *for* $f \in F$, *and*

(v) $\|RP(x)\| \leqq R(x)$ *for* $x \in X$ *with* $|||x||| \leqq 1$.

2.4. Lemma. *Let* X *and* $R : X \to Y$ *be as in Lemma* 2.2, $m = \chi(X)$ *and* $\{x_\xi, \xi < m\}$ *be a dense subset of* X. *Then there is a family of projections* $\{P_\xi, \omega \leqq \xi < m\}$ *with the properties*

$$\|P_\xi\| = |||P_\xi||| = 1, \qquad x_\xi \in P_{\xi+1}(X), \qquad \chi(P_\xi(X)) \leqq |\xi|,$$

$$P_\xi P_\eta = P_\eta P_\xi = P_\eta \qquad \text{for } \omega \leqq \eta < \xi < m,$$

$$\bigcup_{\omega \leqq \eta < \xi} P_{\eta+1}(X) \text{ is dense in } P_\xi(X) \text{ for } \omega < \xi < m,$$

$$\|RP_\xi(x)\| \leqq \|R(x)\| \qquad \text{for } \omega \leqq \xi < m \text{ and } x \in X, \quad \text{with } |||x||| \leqq 1.$$

The proof of this lemma is based on Lemma 2.3 (cf. [1] lemma 6); here the last condition is valid from the weak-weak continuity of $R \mid U$ and the weak*-lower-semicontinuity of the norm.

2.5. Lemma. *Let* X *and* $R : X \to Y^*$ *be as in Lemma* 2.2, *and* $\{P_\xi, \omega \leqq \xi < m\}$ *be a family of projections of* X *as in Lemma* 2.4. *Then for every* $x \in X$ *we have*

$$P_\eta(x) = \|\cdot\|\text{-}\lim_{\xi<\eta} P_\xi(x) \quad \textit{for every limit ordinal } \eta,$$

with $\omega < \eta < m$, *and*

$$x = \|\cdot\|\text{-}\lim_{\xi<m} P_\xi(x).$$

Hence, for every $x \in X$ *and* $\varepsilon > 0$ *the set* $\{\xi : \|P_{\xi+1}(x) - P_\xi(x)\| > \varepsilon\}$ *is finite.*

PROOF. Let $\varepsilon > 0$ and η a limit ordinal, $\omega < \eta < m$. Since $P_\eta(X) = \overline{\bigcup_{\xi<\eta} P_\xi(X)}$, there are $\zeta < \eta$, and $z \in P_\zeta(X)$ such that $\|P_\eta(x) - z\| < \varepsilon/2$. Then for every ξ with $\zeta < \xi < \eta$ we have

$$\|P_\xi(x) - P_\eta(x)\| \leqq \|P_\xi(x) - z\| + \|z - P_\eta(x)\|$$

$$= \|P_\xi P_\eta(x) - P_\xi(z)\| + \|z - P_\eta(x)\| \leqq \|P_\eta(x) - z\| + \|z - P_\eta(x)\| < \varepsilon.$$

We will make use of (a special case of) the following fact due to Dowker (cf. [4], p. 172).

THEOREM. *Let* Ω *be a paracompact space,* $\Phi_1 : \Omega \to \mathbf{R}$ *an upper-semicontinuous and* $\Phi_2 : \Omega \to \mathbf{R}$ *a lower-semicontinuous function such that* $\Phi_1(p) < \Phi_2(p)$ *for every* $p \in \Omega$. *Then there is a continuous function* $f : \Omega \to \mathbf{R}$ *such that* $\Phi_1(p) < f(p) < \Phi_2(p)$ *for every* $p \in \Omega$.

2.6. LEMMA. *Let* M *be a metric space,* $p_0 \in M$ *and* $\Phi : M \to \mathbf{R}$ *a lower-semicontinuous function such that* $\Phi(p) > 0$ *for every* $p \in \Omega$, $p \neq p_0$, *and* $\Phi(p_0) = 0$.

Then there is a continuous function $f : M \to \mathbf{R}$, *such that*

(i) $f(p) > 0$ *for* $p \in M$, $p \neq p_0$,

(ii) $f(p_0) = 0$, *and*

(iii) $f(p) \leqq \Phi(p)$ *for* $p \in M$.

PROOF. We set $\Omega = M \setminus \{p_0\}$. Then, Ω is a paracompact space and hence from Dowker's theorem, there is a continuous function $g : \Omega \to \mathbf{R}$ such that $0 < g(p) < \Phi(p)$ for every $p \in \Omega$. We consider the function $h : \Omega \to \mathbf{R}$ defined by $h(p) = \rho(p, p_0)$ (where $\rho(p, p_0)$ is the distance between p and p_0); then h is continuous, $h(p) > 0$ for every $p \in \Omega$, and $h(p_0) = 0$. Now we set

$$f(p) = \begin{cases} \min\{g(p), h(p)\} & \text{if } p \in \Omega, \\ 0 & \text{if } p \neq p_0. \end{cases}$$

The function f satisfies the conclusion of the lemma.

2.7. Lemma. *Let M be a compact metric space, $p_0 \in M$ and $\Phi : M \to \mathbf{R}$ a lower-semicontinuous function such that $\Phi(p) > 0$ for every $p \in M$ with $p \neq p_0$, and $\Phi(p_0) = 0$. Then there is a one-to-one, continuous function $s : M \to c_0$ such that*

$$\| s(p) \|_{c_0} \leqq \Phi(p) \qquad \text{for } p \in M$$

(where we consider the weak topology on c_0).

Proof. From Lemma 2.6 there is a continuous function $f : M \to \mathbf{R}$ such that $f(p) > 0$ for every $p \neq p_0$, $f(p_0) = 0$, and $f(p) \leqq \Phi(p)$ for every $p \in M$.

Let $g_1, g_2, \cdots$ be a sequence of real continuous functions on M separating points of M, with $\| g_n \| = 1$, $n = 1, 2, \cdots$. We set

$$s(p) = \left(f(p), f(p) g_1(p), \frac{f(p) g_2(p)}{2}, \cdots \right) \qquad \text{for } p \in M.$$

Then it is easy to verify, using Proposition 1.2, that s satisfies the required conclusion.

2.8. Lemma. *Let X and $R : X \to Y^*$ be as in Lemma 2.2. Then there is a set Γ and a function $s : R(U) \to c_0(\Gamma)$ weak*-weak continuous, one-to-one and $\| sR(x) \|_{c_0(\Gamma)} \leqq \| x \|$ for every $x \in U$.*

Proof. The proof will be by induction on the density character of X.

(1) $\chi(X) = \aleph_0$. Then U is weakly compact and metrizable. Hence $R(U)$ is weak* compact and metrizable. We set

$$\Phi(p) = \inf\{ \| x \| : x \in U, R(x) = p \} \qquad \text{for } p \in R(U).$$

Then

(i) $\Phi(p) \geqq 0$, and $\Phi(p) = 0$ if and only if $p = 0$. Indeed we have $\Phi(0) = 0$, because $R(0) = 0$. Now let $p \neq 0$. Since $R \mid U$ is weakly-weakly* continuous and U is weakly closed, we have that the set $R^{-1}(\{p\}) \cap U$ is weakly closed, and hence norm-closed. The element 0 does not belong to $R^{-1}(\{p\}) \cap U$ and so the distance of 0 from this set is strictly positive, that is $\Phi(p) > 0$.

(ii) The function Φ is lower-semicontinuous (we omit the simple proof). Consequently it follows from Lemma 2.7 that there is a function $s : R(U) \to c_0$ weakly*-weakly continuous, one-to-one, and

$$\| s(p) \|_{c_0} \leqq \Phi(p) \qquad \text{for } p \in R(U),$$

i.e., $\| sR(x) \|_{c_0} \leqq \| x \|$ for $x \in U$.

(2) Let $\chi(X) = m > \aleph_0$. We suppose that the conclusion is valid for every infinite cardinal less than m.

From Lemmas 2.4 and 2.5 there is a family of projections $\{P_\xi : \omega \leqq \xi < m\}$ such that

$$\chi(P_\xi(X)) \leqq |\xi|, \qquad \|P_\xi\| = |||P_\xi||| = 1,$$

for every $x \in X$ and $\varepsilon > 0$ the set $\{\xi : \|P_{\xi+1} - P_\xi(x)\| > \varepsilon\}$ is finite, $P_\xi(x) = \|\cdot\|$-$\lim_{\zeta<\xi} P_\zeta(x)$ for every $x \in X$ and $\omega < \xi < m$, ξ limit ordinal, and

$$x = \|\cdot\|\text{-}\lim_{\xi<m} P_\xi(x).$$

Let U_ω be the $|||\cdot|||$-unit ball in $P_\omega(X)$. Then U_ω is $\|\cdot\|$-weakly compact and $P_\omega(U) \subset U_\omega$, since $|||P_\omega||| = 1$. Also for $\omega \leqq \xi < m$, let $U_{\xi+1}$ be the $|||\cdot|||$-unit ball in the space $\frac{1}{2}(P_{\xi+1} - P_\xi)(X)$. Then $U_{\xi+1}$ is $\|\cdot\|$-weakly compact and $\frac{1}{2}(P_{\xi+1} - P_\xi)(U) \subset U_{\xi+1}$, because $|||\frac{1}{2}(P_{\xi+1} - P_\xi)||| \leqq 1$.

We apply the induction hypothesis

$$\text{for } P_\omega(X), \quad R_\omega = R \mid P_\omega(X), \qquad \text{and also}$$

$$\text{for } \tfrac{1}{2}(P_{\xi+1} - P_\xi)(X), \quad R_{\xi+1} = R \mid \tfrac{1}{2}(P_{\xi+1} - P_\xi)(X), \qquad \omega \leqq \xi < m,$$

and let $s_\omega : R(U_\omega) \to c_0$,

$$s_{\xi+1} : R(U_{\xi+1}) \to c_0(\Gamma_{\xi+1}), \qquad \omega \leqq \xi < m$$

be the resulting functions.

We consider the discrete union $\Gamma = \mathbf{N} \cup (\bigcup_{\omega \leqq \xi < m} \Gamma_{\xi+1})$ and define for $x \in U$,

$$\upsilon(x)(n) = [s_\omega R P_\omega(x)](n) \qquad \text{for } n \in \mathbf{N}.$$

$$\upsilon(x)(\gamma) = [s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1} - P_\xi))(x)](\gamma) \qquad \text{for } \gamma \in \Gamma_{\xi+1}.$$

Then $\upsilon(x)$ belongs in $c_0(\Gamma)$. Indeed, let $\varepsilon > 0$. Then the set

$$\{n \in \mathbf{N} : |\upsilon(x)(n)| > \varepsilon\} = \{n \in \mathbf{N} : |[s_\omega R P_\omega(x)](n)| > \varepsilon\}$$

is finite, since $s_\omega R P_\omega(x) \in c_0$.

Also, for every ξ, $\omega \leqq \xi < m$ the set

$$\{\gamma \in \Gamma_{\xi+1} : |\upsilon(x)(\gamma)| < \varepsilon\} = \{\gamma \in \Gamma_{\xi+1} | [s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1} - P_\xi))(x)](\gamma)| > \varepsilon\}$$

is finite, since $s_{\xi+1} R(\frac{1}{2}(P_{\xi+1} - P_\xi))(x) \in c_0(\Gamma_{\xi+1})$.

The set

$$\{\xi \geqq \omega : \| s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1} - P_\xi))(x) \|_{c_0(\Gamma_{\xi+1})} > \varepsilon\}$$

is finite, since by induction hypothesis we have

$$\| s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1} - P_\xi))(x) \|_{c_0(\Gamma_{\xi+1})} \leqq \tfrac{1}{2} \| P_{\xi+1}(x) - P_\xi(x) \|$$

and the set $\{\xi : \| P_{\xi+1}(x) - P_\xi(x) \| > 2\varepsilon\}$ is finite. Consequently, indeed $\upsilon(x) \in c_0(\Gamma)$.

We observe that the set $\upsilon(U)$ is bounded with respect to the norm, as a subset of $c_0(\Gamma)$. (In fact, U is weakly compact and hence $\|\cdot\|$-bounded. Let θ be a $\|\cdot\|$-bound of U. Then for $x \in U$, $n \in \mathbf{N}$ and $\gamma \in \Gamma_{\xi+1}$ we have

$$| \upsilon(x)(n) | \leqq \| s_\omega R P_\omega(x) \|_{c_0} \leqq \| P_\omega(x) \| \leqq \| x \| \leqq \theta, \qquad \text{and}$$

$$| \upsilon(x)(\gamma) | \leqq \| s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1}(x) - P_\xi(x))) \|_{c_0(\Gamma_\xi)} \leqq \tfrac{1}{2} \| P_{\xi+1}(x) - P_\xi(x) \| \leqq \| x \| \leqq \theta;$$

hence $\| \upsilon(x) \|_{c_0(\Gamma)} \leqq \theta$ for $x \in U$.)

The function υ is weakly continuous, because $R \mid U$ is weakly-weakly* continuous, s_ω and $s_{\xi+1}$ are weakly*-weakly continuous, P_ω and $\frac{1}{2}(P_{\xi+1} - P_\xi)$ are weakly continuous, and the weak topology on the bounded subsets of $c_0(\Gamma)$ is identified with the pointwise convergence topology (1.2). The function υ has the following property: $\upsilon(x) = \upsilon(y)$ if and only if $R(x) = R(y)$. Indeed, let $x \neq y$ and $R(x) = R(y)$; then we have

$$\left\| R P_\xi \left(\frac{x - y}{|||x - y|||} \right) \right\| \leqq \left\| R \left(\frac{x - y}{|||x - y|||} \right) \right\|,$$

and hence $RP_\xi(x) = RP_\xi(y)$. From the definition of υ, it follows that $\upsilon(x) = \upsilon(y)$. Conversely, let $\upsilon(x) = \upsilon(y)$. Then we have

$$s_\omega R P_\omega(x) = s_\omega R P_\omega(y) \qquad \text{and}$$

$$s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1}(x) - P_\xi(x))) = s_{\xi+1} R(\tfrac{1}{2}(P_{\xi+1}(y) - P_\xi(y))),$$

and hence

$$R P_\omega(x) = R P_\omega(y) \qquad \text{and}$$

$$R(P_{\xi+1}(x) - P_\xi(x)) = R(P_{\xi+1}(y) - P_\xi(y)),$$

because s_ω and $s_{\xi+1}$ are one-to-one.

It follows that $RP_\omega(x - y) = 0$, and

$$R P_{\xi+1}(x - y) = R P_\xi(x - y) \qquad \text{for every } \omega \leqq \xi < m,$$

and hence that $RP_\xi(x - y) = 0$ and every $\omega \leqq \xi < m$.

It now follows that $R(\frac{1}{2}(x-y)) = \lim_{\xi<m} RP_\xi(\frac{1}{2}(x-y)) = 0$, i.e., that

$$R(x) = R(y).$$

We define $s : R(U) \to c_0(\Gamma)$ by $sR(x) = \upsilon(x)$. It follows for the properties of υ that the function s is well-defined and one-to-one.

It is easy to verify that the function s is weakly*-weakly continuous and

$$\|sR(x)\|_{c_0(\Gamma)} \leqq \|x\| \qquad \text{for } x \in U.$$

2.9. THEOREM. *Let X be a Banach space, K a weakly compact subset of X such that the closed linear hull of K is X. Also, let Y be a Banach space, and let $T : X^* \to Y^*$ be a linear operator continuous with respect to the norm and to the weak* topologies. Then there are a set Γ and a function $s : T(B_{X^*}) \to c_0(\Gamma)$ weakly*-weakly continuous, one-to-one (and hence, a weak*-weak topological embedding) such that*

$$\|sT(x^*)\| \leqq \sup_{x\in K} |x^*(x)| \qquad \textit{for } x^* \in B_{X^*}$$

(where B_{X^*} denotes the unit ball of X^*).

PROOF. We consider K with the weak topology. As is known (cf. [3], p. 146, prop. 1) we have that the space $C(K)$ is weakly-compactly generated. (The same result follows easily from Lemma 2.8 as well, using the Stone–Weierstrass theorem.) We consider the linear operator

$$\Lambda : X^* \to C(K)$$

defined by $\Lambda(x^*) = x^* | K$. Since K spans X, the operator Λ is one-to-one. By the Alaoglou–Bourbaki theorem, it follows that $\Lambda(B_{X^*})$, with respect to the point-wise convergence topology, is compact. Also $\Lambda(B_{X^*})$ is bounded in the uniform norm of $C(K)$. Therefore, by the Grothendieck Theorem 1.3, $\Lambda(B_{X^*})$ is a weakly compact subset of $C(K)$. It follows that the function $\Lambda | B_{X^*}$ is weakly*-weakly continuous. In $\Lambda(X^*)$ we define a second norm by

$$|||\Lambda(x^*)||| = \|x^*\|.$$

Then the $|||\cdot|||$-unit ball U is weakly compact because $U = \Lambda(B_{X^*})$. The operator $\Lambda : X^* \to (\Lambda(X^*), |||\cdot|||)$ is an isometry. We set

$$R = T\Lambda^{-1} : \Lambda(X^*) \to Y^*.$$

Thus the hypothesis of Lemma 2.8 is satisfied (for $\Lambda(X^*)$ and for $\|\cdot\|$ the uniform norm of $C(K)$ restricted to $\Lambda(X^*)$). Hence, there are a set Γ and a function $s : R(U) = T(B_{X^*}) \to c_0(\Gamma)$ weak-weak* continuous, one-to-one, and

$$\|sR(f)\|_{c_0(\Gamma)} \leqq \|f\| \qquad \text{for } f \in U,$$

hence, $\|sT(x^*)\|_{c_0(\Gamma)} \leqq \sup_{x \in K} |x^*(x)|$ for $x^* \in B_{X^*}$.

2.10. Corollary. *Let X be a weakly compactly generated Banach space and Y a Banach space such that Y embeds isomorphically in X. Then the weak* compact subsets of Y* are Eberlein compact.*

Proof. It is enough to prove that there is $\varepsilon > 0$ such that the set

$$B^{\varepsilon}_{Y^*} = \{y^* \in Y^* : \|y^*\| \leqq \varepsilon\}$$

is Eberlein compact. Let $T : Y \to X$ be an isomorphic embedding. Then $T^* : X^* \to Y^*$ is onto, continuous with respect to the norm, to the weak* topologies.

Since T^* is onto, it follows from the open mapping theorem that there is $\varepsilon > 0$ such that $B^{\varepsilon}_{Y^*} \subset T^*(B_{X^*})$.

From Theorem 2.9 there are a set Γ and a function

$$s : T^*(B_{X^*}) \to c_0(\Gamma)$$

weak*-weak continuous and one-to-one.

Consequently, $B^{\varepsilon}_{Y^*}$ is Eberlein compact.

2.11. Corollary (Benyamini-Rudin–Wage [2]). *Let Ω be an Eberlein compact space, S a Hausdorff topological space, and $f : \Omega \to S$ continuous and onto. Then S is an Eberlein compact.*

Proof. The space $C(\Omega)$ is weakly compactly generated and $C(S)$ embeds isomorphically in $C(\Omega)$. Hence it follows from Corollary 2.10 that the unit ball of $C(S)$ with respect to the weak* topology is Eberlein compact. The space S embeds in the unit ball of $C(S)^*$. Thus S is Eberlein compact.

2.12. Remark. The existence of a bounded linear operator $T : X^* \to c_0(\Gamma)$, such that T is one-to-one and weakly*-weakly continuous implies that X is W.C.G. This fact, together with Rosenthal's well-known example (in [6]) of a non-W.C.G. subspace of a W.C.G. space, explains why Theorem 2.9 cannot be linear.

2.13. Remark. Recently M. E. Rudin has proved that if a space is normal, countably paracompact, and homeomorphic to a subspace of $c_0(\Gamma)$ in its weak topology, then every *perfect* (i.e. such that its inverse set function maps compact sets to compact sets) image of this space is homeomorphic with a subspace $c_0(\Gamma)$ in its weak topology. It is interesting in this connection that in the proof of our basic theorem 2.9 we need an interpolation property of semicontinuous functions that in fact is equivalent to the property of normality together with countable paracompactness (e.g. see p. 172 in [4]); this raises the question whether there can be some more essential connection between our methods, and results established by M. E. Rudin.

Added in proof. It was pointed out by the referee that another proof of the Benyamini–Rudin–Wage Theorem appears in: S. P. Gulko, *On properties of subsets of Σ-products*, Soviet Math. Dokl. **18** (1977), 1438.

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